Students' Conceptions of Limits: High Achievers versus Low Achievers

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Abstract: *Learning an advanced mathematical concept, limits of functions in this case, is not a linear development equal for all learners. Intentions and abilities influence students' learning paths and results. Students' learning developments of limits were studied in terms of concept images (Tall & Vinner, 1981) in the sense that their actions, such as problem solving and reasoning, were considered traces of their mental representations of concepts. High achievers' developments were compared to low achievers' developments to for the duration of a semester to reveal differences and similarities.*

1. Introduction

Students learning limits of functions perceive and treat limits differently. Embracing limits of functions demands certain abstraction skills from the students. There are several cognitively challenging issues to deal with, such as understanding the quantifiers' roles in the formal definition or linking formally expressed theory to everyday problem solving. Students accept different levels of understanding as they have different priorities and abilities. Each student has his or her own conceptual development during a course and the question is; how do high achieving students' conceptual developments differ from low achieving students' developments?

A study on students' conceptual development of limits of functions was conducted at a Swedish university (Juter, 2006a) with the purpose to describe students' developments as they learned limits in a basic calculus course. The results imply differences in high achieving and low achieving students' work with limits, but also a lack of differences at some points as will be discussed further on in this article.

2. A model of concept representations

Tall (2004) has introduced three worlds of mathematics to distinguish different modes of mathematical thinking, with the purpose to "gain an overview of the full range of mathematical cognitive development" (Tall, 2004, p. 287). The theory of the three worlds emphasizes the construction of mental representations of concepts and has emerged from several theories on concept development, such as Sfard's (1991) work on encapsulation of processes to objects and Piaget's abstraction theories (Tall, 2004). The three worlds are

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somewhat hierarchical in the sense that there is a development from just perceiving a concept through actions to formal comprehension of the concept. The first world is called the embodied world and here individuals use their physical perceptions of the real world to perform mental experiments to build mental conceptions of mathematical concepts. The mental experiments can be children's categorisations of real-world objects, such as an odd number of items or, later, students' explorations of intuitive perceptions of limits of functions. The second world is called the proceptual world. Here individuals start with procedural actions on mental conceptions from the first world, as counting, which by using symbols become encapsulated as concepts. The symbols represent both processes and concepts, for example counting and number or addition and sum. The symbols, together with the processes and the concepts, are called procepts (Gray & Tall, 1994) and are used dually as processes and concepts depending on the context. The third world is called the formal world and here properties are expressed with formal definitions as axioms. There is a change from the second world with connections between objects and processes to the formal world with axiomatic theories comprising formal proofs and deductions. Individuals go between the worlds as their needs and experiences change and mental representations of concepts are formed and altered.

Not all mathematical concepts can be regarded as an object and a process, e.g. a circle or an equivalence class that are both pure objects, though in limits this duality is very obvious. Limits can be handled through an explorative approach with tables of function values and graphs from the beginning and later as symbolically expressed entities. Learning limits of functions demands leaping between operational and static perceptions (Cottrill et al., 1996). There is a challenge in understanding the termination of an infinite procedure as a finite object, such as $\lim_{x\to\infty}\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}=1$. It is important to reach all significant stages and be able to change between the different stages. Only then can an individual fully understand the concept if understanding of a mathematical concept is defined as Hiebert and Carpenter did (1992), i.e. to be something an individual has achieved when he or she can handle the concept as part of a mental network. The more connections between the mental representations, the better the individual understands the concept (Dreyfus, 1991; Hiebert & Carpenter, 1992).

In an attempt to create a model for concept development, I have used theories about concept images (Tall & Vinner, 1981; Vinner, 1991) as a complement to the theory of the three worlds. A concept image for a concept is an individual's total cognitive representation for that concept. The concept image comprises all representations from experiences linked to the concept, of which there may be several sets of representations constructed in different contexts that possibly merge as the individual becomes more mathematically mature. Multiple representations of the same concept can co-exist if the individual is unaware of the fact that they represent the same concept. Possible inconsistencies may remain unnoticed if the inconsistent parts are not evoked simultaneously. Concept images are created as individuals go through the developments represented by the three worlds. The model in Figure 1 shows how part of a concept image can be structured as I consider it. The three types of symbols used each represent a concept at the stage of one of Tall's three worlds, as described in the figure. The concepts can be, for instance, geometric series, derivatives of polynomials, definitions of derivatives and limits of functions, theorems, proofs, and examples of topics of related concepts. More links and more representations of concepts exist around the formal world representations of concepts. There are also parts that are not very well connected to other parts. This situation can occur when individuals use rote learning as they try to cope with mathematics. Students who are unable to encapsulate processes as objects or take the step from procepts to a strictly formalistic exposition can use rote learning as a substitute.

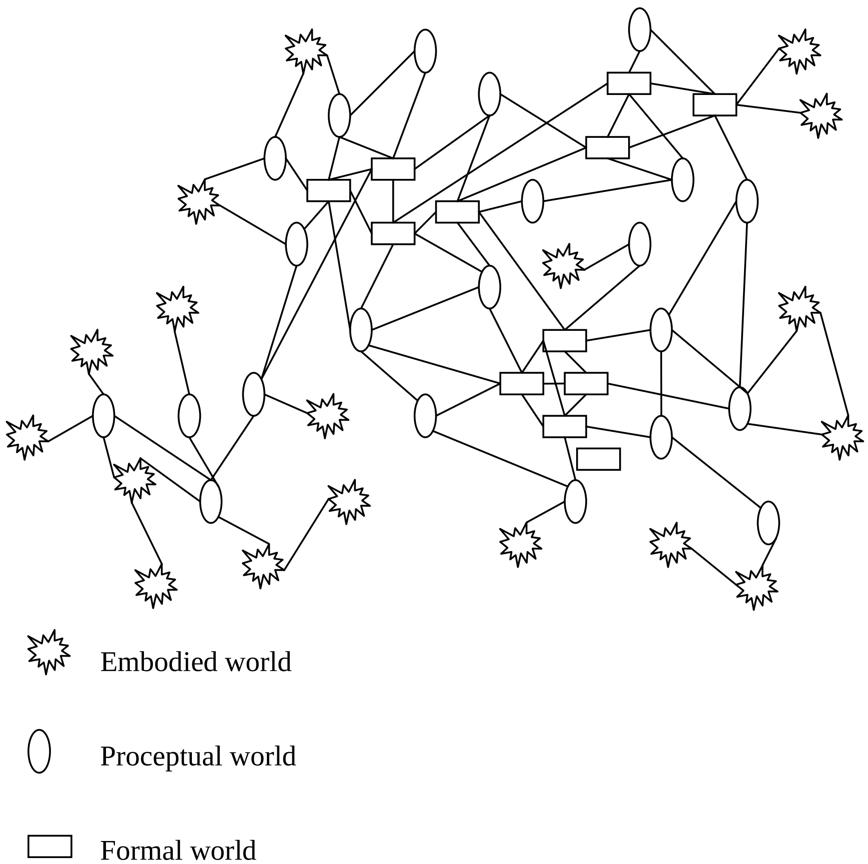

Figure 1. Model of part of a concept image at one time. Each node is a representation of a concept at one of the different stages of Tall's three worlds.

Mental representations can be depicted in terms of *topic areas* as a complement to the levels of abstraction shown in Figure 1. A topic area refers to the areas of mathematics with components of a certain topic, e.g. 'functions' or 'limits'. The components are such nodes as those in Figure 1. The sizes of topic areas vary according to what context they appear in, for instance large areas such as 'functions', or smaller areas such as 'polynomial functions'. The classification in topic areas means sub-topic areas at several levels. A component in one topic area can in itself be a topic area. Weierstrass's limit definition belongs to the topic area of 'limit', as do 'limits of rational functions' and the symbols used to express limits. The symbols also belong to the topic areas 'derivatives' and 'continuity'. Topic areas overlap this way as illustrated by the simplistic model in Figure 2.

If a concept is represented in more than one topic area in a concept image and the topic areas the representations belong to are disjoined, then inconsistencies may occur in the way aforementioned. Inconsistencies can appear within a topic area as well, but they are easier to detect due to the relatedness of the topic. The development of concept images never ends and the mental representations generate a dynamical system linked together at various levels.

An example of a topic area, marked by a wider contour line in Figure 2 represents the topic area ‘limits’. It comprises a marked oval component representing the limit definition, which is also part of the topic areas ‘derivatives’, TA2, and ‘continuity’, TA3. The black rectangular component represents the definition of derivatives. The figure only shows some nodes in each topic area to describe the structures of the complicated relations. There are, in most real cases, more nodes linked in more intricate constellations.

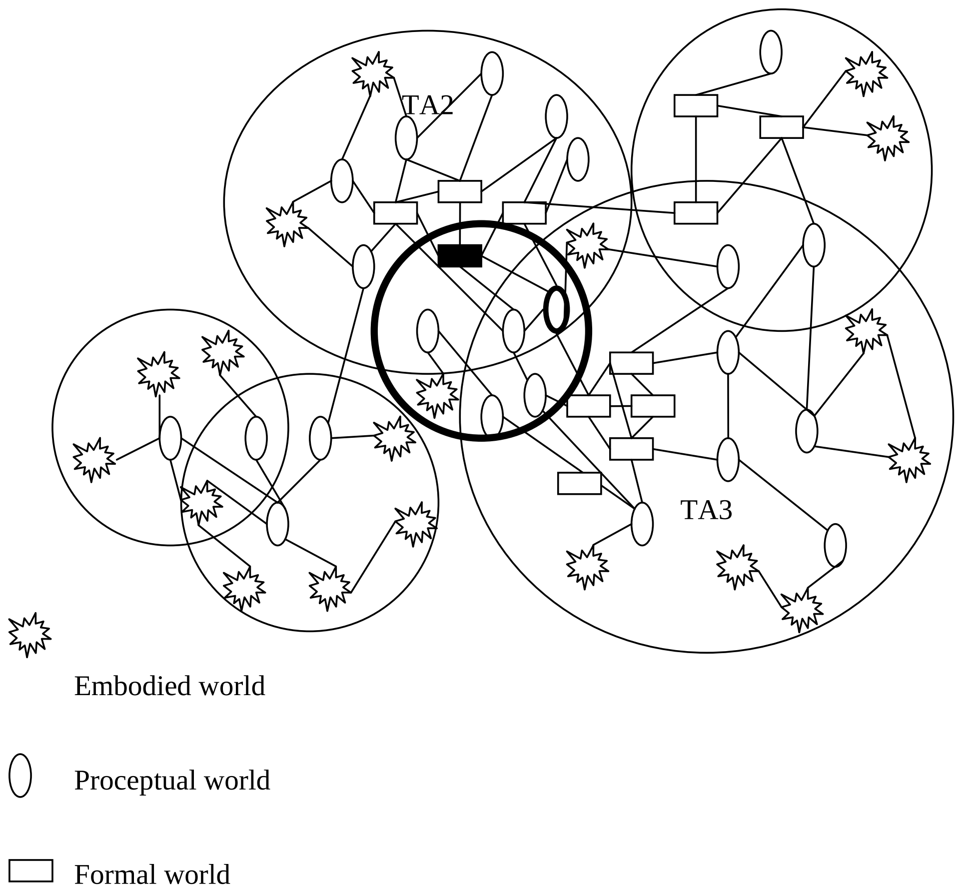

Figure 2. Topic areas and components with links in a model of a concept image. The marked part is the topic area ‘limits’. TA2 and TA3 represent the topic areas ‘derivatives’ and ‘continuity’ respectively.

Concept images change on account of outer and inner stimuli, such as discussions, thoughts and problem solving, and a model such as the one in Figure 2 is hence in constant change. It is nevertheless a tool suitable for describing students' concept developments of limits of functions.

3. The empirical study

This section describes the sample of students studied and the course they were enrolled in, followed by an outline of methods and instruments used.

3.1 The students and the course

There were 112 students participating in the study, of these, 33 were female. The students were aged 19 and up. They were enrolled in a first level university course in mathematics that was divided into two sub-courses. Both of them dealt with calculus and algebra and were given over 20 weeks full time (10 weeks for each course). The students had two lectures (the whole group with one lecturer) and two sessions for task solving (in sub-groups of 30 students with a teacher in each sub-group) three days per week. Each lecture and session lasted 45 minutes. Thus the total teaching time for each course was 90 hours.

The notion of limits of functions was presented in the first course before derivatives. The lectures and sessions dealing with limits are outlined here to describe the students' first encounter with limits of functions at university level. On the first lecture on limits, the lecturer followed the textbook presenting formal definitions and theorems on indefinite and definite limits of functions and limits of monotonic functions as x tends to infinity (for functions depending on x). The textbook has an intuitive approach in the initial pages of the book, but the exposition becomes strictly formal after that. On the following task solving session, the students in the group were reluctant to go up to the black board to solve tasks, and the teacher ended up solving seven of ten tasks for that session. Students tried to solve three of the seven tasks before the teacher solved them. The students said that absolute values confused them and that was also one of the problems with the tasks.

The second lecture dealt with standard limit values and some proofs were rapidly presented (some comments on the speed were whispered among the students). The number e was introduced and so were ε-δ definitions as x tends to a number. Continuity was then presented with some following theorems. Parts of the proofs were omitted. The lecturer kept on following the textbook to help the students follow his reasoning. The second task solving session was very similar to the first where the teacher solved most of the tasks. The triangle inequality was discussed. A question of whether a function can have several limits was posed and answered.

In the third lecture the lecturer continued to prove theorems from last lecture. Trigonometric formulas were repeated from lectures before limits were taught. Some theorems were made plausible through pictures. Derivatives were introduced. The lecturer said that derivatives and integrals were what the course is all about. On the following task solving session, there were many questions from the students about theorems and definitions and how to use the theories they had met. Continuity and monotonic functions were particularly discussed. The rest of the session was about derivatives.

The following lectures and sessions dealt with derivatives and integrals. Limits were taught again in the second course in different settings such as integrals and series. The first course had a written exam and the second had a written exam followed by an oral one. The marks awarded were *IG* for not passing, *G* for passing and *VG* for passing with a good margin.

3.2 Methods

Different methods were used to collect different types of data, such as students' solutions to limit tasks and responses to attitudinal queries. The sets of data were collected at different stages in the students' developments. The instruments used were designed to take those differences into account. The limit tasks were of increasing difficulty and the attitudinal part was mainly in the beginning of the semester. The students were confronted with tasks at five times during the semester, called *stage A* to *stage E*.

The students got a questionnaire at stage A in the beginning of the semester. It contained easy tasks about limits and some attitudinal queries. The scope of these and subsequent tasks is described in the instruments section. The students were also asked about the situations in which they had met the concept before they started their university studies. The attitudinal data are not presented in this article.

After limits had been taught in the first course, as described in the former section, the students received a second questionnaire at stage B, with more limit tasks at different levels of difficulty. The aim was for the students to reveal their habits of calculating, their abilities to explain what they did, and their attitudes in some areas. The students were asked if they were willing to participate in two individual interviews later that semester. Thirty-eight students agreed to do so; of these, 18 students were selected for two individual interviews each. The selection was done with respect to the students' responses to the questionnaires so that the sample would as much as possible resemble the whole group. The gender composition of the whole group was also considered in the choices.

The first session of interviews was held at stage C in the beginning of the second course. Each interview was about 45 minutes long. The students were asked about definitions of limits, both the formal one from their textbook and their individual ways to define a limit of a function. They also solved limit tasks of various types with the purpose to reveal their perceptions of limits and commented on their own solutions from the questionnaires to clarify their written responses where it was needed.

The students received a third questionnaire at the end of the semester, at stage D. It contained just one task. Two fictional students' discussion about a problem was described. One reasoned incorrectly and the other one objected and proposed an argument to the objection. The students in the study were asked to decide who was correct and why.

A second interview was carried through at stage E after the exams. Each interview lasted for about 20 minutes. Of the 18 students, 15 were interviewed at this point. The remaining three students were unable to participate for various reasons. The students commented on the last questionnaire and, linked to that, the definition was scrutinized again. The quantifiers *for every* and *there exists* in the ε - δ definition were discussed thoroughly. Of the 15 interviewed students, three low achievers and three high achievers were selected for comparisons as indicated in Table 1.

Table 1: Students' marks

Name	**Mark**
Martin	G1/G2
Tommy	G1/G2
Anna	G1/G2
Julia	VG1/VG1
Dennis	VG1/VG1
Emma	VG1/VG1

Field notes were taken during the students' task solving sessions and at the lectures when limits were treated to give a sense of how the concept was presented to the students and how the students responded to it. Tasks and results from other parts of the study are described in more detail in other articles (Juter, 2005a-2006c).

3.3 Instruments

The students solved some easy tasks about limits of functions at stage A, such as the following example:

Example 1: f(x) = $\frac{x^2}{x^2+1}$. What happens with f(x) if x tends to infinity?

The tasks did not mention limits per se, but were designed as a means to explore if the students could investigate functions with respect to limits.

At stage B the tasks were more demanding. Some of the tasks were influenced by Szydlik (2000) and Tall and Vinner (1981). Three tasks had the following structure:

Example 2: a) Decide the limit: $\lim_{x\to\infty}\frac{x^3-2}{x^3+1}$.

b) Explanation.

c) Can the function f(x) = $\frac{x^3-2}{x^3+1}$ attain the limit value in 2a?

d) Why?

Example 2 is what I regard to be a routine task. There were also non-routine tasks. A solution to a task was presented to the students. It could be incomplete or wrong and the students were to make it complete and correct. There were two such tasks. The students were also asked to formulate a definition of a limit, not necessarily the one in their textbooks.

At stage C, which was the first set of interviews, the students were asked to comment on statements very similar to those used by Williams (1991) in a study about students' models of limits. The statements the students commented on are the following (translation from Swedish):

1. A limit value describes how a function moves as *x* tends to a certain point.
2. A limit value is a number or a point beyond which a function can not attain values.
3. A limit value is a number which *y*-values of a function can get arbitrarily close to through restrictions on the *x*-values.
4. A limit value is a number or a point which the function approaches but never reaches.
5. A limit value is an approximation, which can be as accurate as desired.
6. A limit value is decided by inserting numbers closer and closer to a given number until the limit value is reached.

The reason for having these statements was to get to know the students' perceptions about the ability of functions to attain limit values and other characteristics of limits. The students were given the statements to have something to compare with their own thoughts. There were other tasks designed to make the students consider the formal definition to clarify what it really says, and tasks about attainability, for example:

Example 3: Is it the same thing to say "For every $\delta > 0$ there exists an $\varepsilon > 0$ such that $|f(x) - A| < \varepsilon$ for every x in the domain with $0 < |x - a| < \delta$" as "For every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - A| < \varepsilon$ for every x in the domain with $0 < |x - a| < \delta$"? What is the difference if any?

As indicated before, at stage D the students got a task with a description of two students arguing over a solution to a task (translation from Swedish):

Example 4: Two students discuss a problem. They do not agree on the solution. The problem discussed is about the following limit: $\lim_{x \to 0} \frac{\cos(x^{-2})}{10000}$.

The student S1 claims that the limit exists and is zero with the following explanation:
I use the definition for limits and write |f(x) – A| where A is the limit. I try A = 0 since I think that is the limit. I get: $\left|\frac{\cos(x^{-2})}{10000} - 0\right| \le \frac{1}{10000} < \frac{1}{9999}$. This is true for all x except x = 0, but

that can never be the case since we then would have zero in the denominator in x^{-2}, so it is true for all x in the domain of f(x) = $\frac{\cos(x^{-2})}{10000}$. This means that if we chose $\varepsilon = \frac{1}{9999}$ for all possible δ with $0 < |x - 0| < \delta$, then the definition is met and the limit is zero.

The student S2 does not agree and claims that if one, for example, chose $\varepsilon = \frac{1}{100000}$ then one can not find a $\delta > 0$ with $|f(x) - A| < \varepsilon$ where $0 < |x - 0| < \delta$ for all x in the domain, that is to say all $x \neq 0$. Therefore, the student S2 claims that $\lim_{x \to 0} \frac{\cos(x^{-2})}{10000}$ has no limit according to the definition of limits.

At stage E, the second set of interviews, the students' written responses to the task at stage D were discussed. *Example 3* was also brought up again in connection with the task at stage D.

4. Typical patterns for the six students' developments of limits

The students' responses to tasks and questions in the questionnaires and interviews have been analysed and categorised. Table 2 shows the typical developments of the students in the categories *High achievers* and *Low achievers* respectively. A developmental portrait was done for each of the 15 interviewed students (published in Juter (2006b)) from which the combined descriptions were drawn.

Table 2: Typical student developments in the two categories through the semester

Stage	High achievers	Low achievers
A	Links limits to prior studies. Solves easy tasks well.	Links limits to prior studies and other topics. Solves easy tasks well.
B	Limits are attainable in problem solving. Solves tasks and explains well. Problems to state a limit definition.	Limits are attainable in problem solving. Solves routine-tasks and explains with some flaws, other tasks problematic. Cannot state a limit definition.
C	Limits are attainable in problem solving and in theory. Prefers statement 3. Problems to state the definition. Can identify the definition with uncertainty about the quantifiers' meanings. Solves tasks fairly well.	Limits are attainable in problem solving but not in theory. Prefers statements 1 and 4. Cannot state the definition. Problems to identify the definition, quantifiers not understood. Problems to solve tasks. Not an actual limit if attainable by the function.
D	Identifies the error.	Problems to identify the error.
E	Can identify the definition. Can explain the quantifiers' meanings.	Problems to identify the definition. Cannot explain the quantifiers' meanings.

There are obvious similarities between the two categories. The quantifiers in the definition caused confusion for all students. There was an opinion among some students that ε and δ in Example 3 at stage C come in pairs and can therefore be placed either way in the example. Mostly high achieving students shoved traces of this conception, which can be explained by the fact that low achieving students had not integrated the theory well enough in their concept images to even identify the definition next to a wrong one. The high achieving students did not have this misunderstanding at stage E as they were able to explain the meaning of the quantifiers. The low achieving students did not understand the quantifiers meaning in the definition for the duration of the course.

The students' problems to connect theory to problem solving became particularly apparent from their difficulties to determine whether limits are attainable for functions or not. Many students interpreted the strict inequalities in the formal definition to say that limits are not attainable. Examples where limits were attainable did not change the low achieving students' beliefs about the definitions' meaning. Some students became frustrated when they saw examples of attainable limits and were asked questions about the definition because they were unable to create a coherent picture of the situation. The students' concept images were divided in disjoint topic areas; one for limits in theory and one for limits in problem solving. High achievers were able to link the two topic areas at the end of the course. Such linking often requires hard work which sometimes involves substantial changes in the students' concept images and the prospect of that makes them disregard inconsistencies and simply see parts that do not cohere with the rest of the concept image as minor exceptions.

Students with positive attitudes to mathematics in general were better limit problem solvers. Most of the high achieving students thought that they had control over the concept of limits, but many of the low achieving students also claimed to have control even if that was not the case. An unjustifiably strong self confidence can prevent students from further work on erroneous or incomplete parts of their concept images.

The students' responses to Example 2 from the instruments section revealed part of the low achieving students' confusion:

4.1 Low achievers:

Martin

a) $\to 1$ b) $\dfrac{1-\frac{2}{x^3}}{1+\frac{1}{x^3}} \to 1$ as $x \to \infty$. c) Yes. d) –

Tommy

a) 1 b) For large *x*, –1 and +2 can be neglected, denominator and numerator are identical and thereby attains one.

c) No. d) Trivial, $x^3 - 2 \neq x^3 + 1$, for the function to attain one requires $x^3 - 2 = x^3 + 1$, logically impossible.

Anna

a) $\rightarrow 1$ b) $\dfrac{x^3\left(1-\dfrac{2}{x^3}\right)}{x^3\left(1+\dfrac{1}{x^3}\right)} = \dfrac{1}{1} \rightarrow 1$. [Arrows showing that $\dfrac{2}{x^3}$ and $\dfrac{1}{x^3}$ tend to zero]

c) No. d) At really large x the value <u>tends</u> <u>to</u> 1 but there will still be a difference between denominator and numerator.

4.2 High achievers:

Julia

a) 1 b) $\dfrac{x^3-2}{x^3+1} = \dfrac{x^3\left(1-\dfrac{2}{x^3}\right)}{x^3\left(1+\dfrac{1}{x^3}\right)}$ i.e. when $x \rightarrow \infty$, –2 and +1 can be neglected since they are so much smaller than *x*.

c) No. d) You cannot insert $x = \infty$.

Dennis

a) 1 b) When *x* becomes very large, –2 and 1 make a smaller difference for the value.

c) No. d) There are different numbers in the denominator and numerator, i.e. they can in fact not be equal and the fraction can therefore not be one.

Emma

a) 1 b) When *x* is very large, the numerator and denominator are in what matters equal. We also have $\dfrac{x^3-2}{x^3+1} = \dfrac{x^3}{x^3} \cdot \dfrac{1-\dfrac{2}{x^3}}{1+\dfrac{1}{x^3}}$ where we can see that for large *x* the expression is approximately equal to one.

c) No. d) $x^3-2 < x^3+1 \ \forall x > 0$. The numerator and denominator are therefore never equal and the fraction is always smaller than one.

High achievers did not use arrows in part a, but the low achievers did. This is one example of students' confusion of limits with function values. The students were uncertain of what happens at the critical point. Anna, in particular, revealed this type of uncertainty as she mixed up her answer in part b. Tommy was also not sure what he believed to be true as he stated two opposite opinions in b and c. He displayed traces of a concept image with multiple incoherent representations. The high achievers did not show this type of confusion, but they were not always clear in their explanations either. Julia, for example, had a vague response to part d where it is impossible to determine whether she reasons correctly or not.

Learning limits requires skills from many mathematical areas. Students need to be able to understand formal expositions, perform algebraic manipulations, understand the meanings of quantifiers and absolute values, which students found problematic, and link theory to their every day problem solving. They also need to find inspiration and reasons to go through the hard work to make the knowledge meaningful in their concept images. High achievers have richer concept images enabling them to create many high quality links and therefore the concept image becomes useful in a variety of situations, which gives the students a broader and clearer view of the topic at hand.

5. Concluding remarks

As could be expected, high achieving students' abstraction abilities were more developed than other students'. The former group was to a much higher degree than the latter able to link theory to problem solving and explain the meaning of, for example, the limit definition. The students were studied during a semester and for that time there were similarities of the high achieving students' developments with the historical development of limits that the other students did not reveal. The similarities were mainly linked to abstraction and formality as the students started with an operational approach with a focus on problem solving rather than theory and then gradually understood the links between theory and problem solving.

There were no clear patterns of students' mental representations of limits as exact values or approximations, limits as objects or processes, and limits as attainable or unattainable for functions. Of the 15 students interviewed, only two showed a coherent trace of their concept images. Both students were high achievers. The lack of patterns in all students' concept images, particularly in the high achievers', points to the complex nature of limits and the challenge to teach and learn limits.

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