

## CONCEPT DEFINITION AND CONCEPT IMAGE

In the case of equations

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**Abstract:** *The purpose of this study is to analyse what kind of conceptions secondary school teachers in mathematics have about equations and how these conceptions are related to the formal definition of the concept of equation. Data was gathered by interviews and questionnaires. Both newly graduated and experienced secondary school teachers were participated in this study. The phenomenographic research method in order to analyse research outcomes was applied in the investigation. From a phenomenographic analysis of the interview transcripts I found that some patterns could be identified in them and the three qualitatively distinct categories of description about equations could be discerned among the teachers' conceptions. The research results indicated that equations were apprehended as a procedure, as an answer and as a 'rewritten' expression.*

Keywords: conception, equation, mathematics teacher

### INTRODUCTION

In this paper I discuss questions about mathematical knowledge, a theory of learning of mathematical concepts and a theory of concept definition and concept image. I will illustrate the theoretical framework with an empirical study. The investigation demonstrates what kind of conceptions the mathematics teachers have of the mathematical concept 'equation'.

It is common to use the terms 'procedural knowledge' and 'conceptual knowledge' to denote a distinction between two forms of mathematical knowledge (Hiebert & Lefevre 1986). Procedural knowledge refers to computational skills and knowledge of procedures for identifying mathematical components, algorithms and definitions. Conceptual knowledge refers to knowledge of the underlying structure of mathematics. It is characterised as knowledge rich in relationships and includes the understanding of mathematical concepts, definitions and fact knowledge. Both procedural and conceptual knowledge are considered as necessary aspects of mathematical understanding (Hiebert & Lefevre 1986). Thus teaching of mathematical understanding must include teaching in both procedural and conceptual knowledge.

Sfard (1991) has analysed the development of various mathematical concepts, definitions and representations from an historical and a psychological perspective. Sfard's analysis has shown that abstract notations such as a rational number, function etc. can be conceived in two fundamentally different ways: operationally as processes or structurally as objects. According to Sfard the process of concept learning includes three stages: 1) *interiorization*: a learner gets acquainted with a concept and performs operations or processes on mathematical objects, 2) *condensation*: a learner has an increasing capability to alternate between different representations of a concept and 3) *reification*: a learner can conceive of the mathematical concept as a complete, "fully-fledged" object. At the stage of reification the new entity is detached from the process which produced it and the concept begins to receive its meaning as a member of certain category. The first two stages represent the operational aspect of mathematical notation and the last stage its structural aspect. According to Sfard (1991) the structural conception of a mathematical notation is static whereas the operational conception is dynamic and detailed. To understand the structural aspect of a mathematical concept is difficult for most people because a person must pass the ontological gap between the operational and structural stage. Sfard (1991, 3) distinguishes between the words "concept" and "conception". According to Sfard the term concept represents the mathematical, formal side of the concept and conceptions the private side of the concept.

This seems to draw on Tall's and Vinner's (1981) theory of concept image and concept definition. They suggest that when we think of a concept something is evoked in our mind. Often these images do not necessarily relate to a concept definition even if the concept is well defined theoretically. The collection of conceptions is called the concept image. In Tall's and Vinner's theory, concept image is the whole cognitive structure that is associated with the concept.

Küchemann's (1981) investigation has showed that a very small percentage of the 13- to 15-year-old pupils were able to consider the letter as a generalised number and even fewer were able to interpret letters as variables. The majority of students either treated letter as concrete objects or ignored them at whole.

Pupils encounter the equals sign from an early stage. However, the limited view of equal sign results difficulties for students to understand mathematical notations. Outside in the arithmetic classroom, the equals sign is often used in the meaning "it is "( as in MATH = DIFFICULT), " it gives " (as in HARD WORK = SUCCESS) ". It is also very likely that students use the equal signs in connected to a question such as "find the solution to the following equation  $5x -$

$2 = 10$  or in the meaning “ a left-to-right directional signal” or “ to do something”. In algebra, this meaning of “ = ” is however not correct, because there usually are no question on one side of the equals sign, and no “answer” on the other (Kieran 1981). Findings of Wagner et al (1984) show that many algebra students have a operational interpretation of a algebraic expressions, because they try to add “ = 0 “ to any expressions they where asked to simplify.

In contrast to the concepts in everyday life, mathematical concepts are well defined. Concept definitions are the body of words used to designate the concepts. For example the definition of the ‘equation’ is a mathematical statement of the following form:

*“ equation, a formula that asserts that two expression have the same value; it is either an identical equation (usually called an identity), which is true for any values of the variables, or conditional equation, which is only true for certain values of the variables “* Borowski & Borwein 1989, 194

Formal definitions like this could help us to form a concept image, but they do not guarantee the understanding of the concept. It happens that the moment learners have formed their concept images or their subjective conceptions of mathematical concepts the definitions become unnecessary. Empirical studies also indicate that students have an intention to interpret the mathematical concepts operationally as processes even if the concepts in teaching of mathematics were introduced structurally, i. e by using definitions (Vinner & Dreyfus 1989; Sfard 1989). The majority of students do not use definitions when solving tasks, because their every day life thought habits take over and they are unaware of the need to consult the formal definitions. In most of the cases referring to just the concept image will be successful. Only non-routine problems like the identification of examples and non-examples of a given concept, problem solving and mathematical proofs, can encourage students to use the formal concept definitions (Vinner 1991). The studies of Linchevsky et al (1992) show that not even students at college or university know what concept definition means.

## **METHOD**

Ten teachers in mathematics at secondary school participated in this study. Five teachers were newly graduated (less than one year’s experience) and five were experienced (between 10 and 32 years’ experience). Data was gathered by interviews and questionnaires. The interviews took place in the schools, where the teachers worked and were recorded. At first, the teachers in the questionnaire including 18 mathematical statements, should identify examples and non-

examples of equations. During the subsequent interview the teachers could develop their thoughts about equations and they had a possibility to explain and motive their answers. The tapes were transcribed and interpreted into the categories of descriptions by applying the phenomenographical research method (Marton & Booth 1997). The concept “conception” is defined in literature in many different ways. In my investigation I use Sfard’s definition of “conception”. “Conception” is a persons’ subjective picture of the concept of equation. (Sfard 1991, 3).

## RESULTS

The categories of description characterize contradictory conceptions that the teachers hold about equations. The solving procedures of equations as well as mathematical symbols are the focus of the teachers’ attention in the first category. The conceptions in the second category indicate that the teachers’ attention is not only focused on the solving procedure but also the arithmetical interpretation of the equals sign. In the third category the teachers’ conceptions about equations have a focus on both the procedures and the rules used in algebra. Significant of the teachers’ conceptions in the three categories is that they reveal a process-oriented view of equations. Now a more careful description of the categories follows.

### Category 1: Equations as a procedure

In the first category the teachers’ conceptions about equations have a focus on the process of solving equations itself and mathematical symbols. *Firstly*, I treat such examples, which were apprehended as non-equations, since the teachers think that it is not possible to solve them, because they consist of too many unknown factors.

For instance the following quotations are examples of how the teachers voice their conceptions in the first category:

*I am a bit unsure if I can solve it... I feel there are three unknown variables. Maybe a formula, but you can’t solve it because you don’t know any values of the variables. (Rita;*

$$2x + 5y = \sqrt{a} )$$

*I felt as if I could draw it, but then we are talking of functions and I became unsure if they could be regarded as equations... I cannot say that  $a=3$  and  $b=2$  in this expression and that it is possible to solve. (Eva;  $a + 5b = 6a + 9,5b$  )*

*I change my mind. I wrote that it is an equation... if the radius is changed, the volume too will be changed and I cannot solve it. So my answer will be no. That is how I feel now.*

(Rita;  $V = \frac{4\pi r^3}{3}$ )

These quotations point out that equations as a mathematical entity are connected with solving procedure. In these remarks the teachers state that 'I cannot solve it'. The reasons for that were e.g. as follows: 'three unknown variables', 'you don't know any values', 'if they [functions] could be regarded as equations...I cannot say that  $a=3...$ ', and 'if the radius is changed, the volume too will be changed'.

Secondly, I consider such examples, which were apprehended as non-equations. They include mathematical symbols like square roots, integrals, derivatives and functions that may prevent the teachers to see the mathematical structure of equations. For instance the following quotations are examples of how the teachers express their conceptions of the mathematical notions.

*How can I be sure of this third root, no it is not so... OK, it is an equation, but I am not quite sure. (Thomas;  $a^b = \sqrt{a\sqrt{a\sqrt{a}}}$ )*

*I associate this to the integral. It is some kind of surface. I don't remember what it is. I feel it is an area under the curve and it can't be an equation. (Maria;  $\int f(x)dx = x^2 + C$ )*

*I want to know what the function itself is called.....so that I can substitute  $y(x)$  by the function and then derive it twice and then solve  $x$ . In principle it is an equation, but I cannot do anything with it. (Eva;  $y''(x) + y(x) = 0,5 \cos 2x$ )*

*It is a function. The equals sign says that it is an equation, but mathematically I don't know, if you can call it an equation. (Jenny;  $f(x) = 2x + 1$ )*

*We press a button on our calculator and it was like this example here... I feel if  $e$  is raised to zero or just the opposite, it will be one. .... That was it. But what it was, I don't know. I felt I began to understand something at university, but now it is gone and I am unsure. (Rita;  $a^{\ln a} = e$ )*

These quotes point out that not only procedures but also mathematical symbols are the focus of the teachers' attention. Many mathematical symbols like 'this third root', 'an area under the curve [primitive function]', 'an integral', and 'e is raised to zero' seem to cause insecurity in the teachers' mind.

Thirdly, I look over such non-examples of equations, which were apprehended as equations, since the teachers think that it is possible to solve them. For instance the following quotations are examples of how the teachers voice their conceptions.

*..it is an equation... I can solve x here. I have a goal... I must have a goal and in the end I can solve x. (Anna;  $x + |x - 3| \geq |x - 1| + 2$ )*

*Yes, I called the example  $x^2 - 5x - 10$  an equation, so this  $(x - \sqrt{2})(x^2 - 9)(5 + x)$  must also be one. (Anna)*

*Yes, I try again if I can get a value of the unknown factor. I can get the value in the exponent. It must be the same x further down. In this case it is an equation. (Anna;  $xe^{7x-1=0}$ )*

These quotes point out that the procedures are the focus of the teachers' attention rather than the mathematical entity itself. The teachers are confused concerning the meaning of many mathematical concepts, which may cause their incorrect classification.

#### Category 2: Equations as an answer

The conceptions in the second category indicate that the teachers' attention is not only focused on the solving procedures but also the arithmetical interpretation of the equals sign. The arithmetical interpretation means that the equals sign is apprehended operationally i.e. followed by an answer. Here I discuss such examples about equations, which were apprehended as non-equations, since the teachers think x is already solved.

The following quotations are examples of how the teachers express their conceptions in the second category of description.

*I apprehend this as an answer.... The value of the unknown factor is already given. That is why it's not an equation. (Maria;  $x = 2$ )*

*No, it is not, because the issue number 7 (that is  $x = 2$  in the questionnaire nr 3) is not an equation... it is only an answer. (Eric;  $e^{x+y} = 1$ )*

*No, it is a rule or a formula. I do not remember what you call it. It is a result of something. (Anna;  $\cos^2 \alpha + \sin^2 \alpha = 1$ )*

These quotes 'it is an answer', 'an expression for what x is' and 'it is a result of something' point out that the equals sign is apprehended 'that is', which is usual in arithmetic. The conception 'The value of the unknown factor is already given' indicates that a teacher may

think that the expression on the left-hand side is a process, which is already performed, whereas the expression on the right-hand side must be an answer.

### Category 3: Equations as a 'rewritten expression'

In the third category the teachers' conceptions about equations have a focus on not only the procedures but also the rules used in algebra.

In this category I treat the statement  $x^2 - y^2 = (x - y)(x + y)$ , which was apprehended as a non-equation, since the teachers think it is a rule or a 'rewritten' expression. The following quotations are examples of how the teachers express their conceptions.

*No, there are no values of figures, you cannot solve  $x$  and  $y$ , it is only an expression, which has been developed. It is a quadratic rule, a conjugate rule. It is just a conjugate rule. (Maria)*

*This is an identity, a conjugate rule. (Thomas)*

*You have rewritten an expression, I don't remember, what you call it. (Eric)*

*You can find here two unknown factors, but if you factorise the left-hand side you receive the right-hand side. Is it always an equation, if there is an equals sign? (Jenny)*

The example evoked the conceptions 'an expression', 'a rewriting of an expression', 'an identity', 'a conjugate rule' and 'a quadratic rule' in the teachers' mind. The quotes point out that 'rules' in algebra are not apprehended as equations.

## **DISCUSSION/CONCLUSIONS**

The research results indicate that the teachers' conceptions about equations appear to be based on procedural knowledge (Hiebert & Lefevre 1986). For some of my teachers 'equation' does not constitute a mathematical statement. The equation concept is for them closely linked to difficulties like the understanding of variables and equality sign, the meaning of the mathematical symbols, the role of the formal definition and the solving procedures. It seems that, the process-object duality of the mathematical notation creates fundamental problems for teachers (Sfard 1991). They have considerable problems in leaving the process level and entering the object level (Sfard 1991).

The research results indicate that some teachers look at algebraic expressions (non-examples of equations) as processes rather than objects in their own right. It is most likely that some

teachers have an operational interpretation of algebraic expressions like  $x^2 - 5x - 10$ ,  $(x - \sqrt{2})(x^2 - 9)(5 + x)$  and  $\sin^2 x + 3\sin x - 4$ . For some teachers the expression like  $x^2 - 5x - 10$  was an example of the equation of the second-degree. Probably, due to the fact that the difference between the concepts 'equation' and 'expression' is not quite clear for some of them. One teacher says about the expression  $\sin^2 x + 3\sin x - 4$ : '*It is only an expression. I don't know what the difference is between an expression and an equation*' (Eric). The research results also indicate that some of the teachers regard an inequality  $(x + |x - 3| \geq |x - 1| + 2)$  and an equation as the same concept, because  $x$  can be solved and the solution to the inequality will be according to them a fixed number.

Students' need to transform algebraic expressions to equations was also presented in a study by Kieran (1983) who found that some of the students could not assign any meaning to the expression  $a + 3$  because the expression lacked an equals sign and right-hand number. Students must, according to Kieran (1992, 392; cf. Sfard 1991) be helped to regard algebraic expressions and equations as objects in their own rights where operations are used, and not as arithmetic operations upon numbers. Kieran also emphasizes that the traditional approach in algebra focuses on procedural issues and various arithmetical techniques, which allow the students to bypass the algebraic symbolism when solving algebraic equations.

Sfard and Linchevski (1994) declare that the process-object duality may affect the development of a complete understanding of the equality relation. The results of this investigation indicate that the equals sign is apprehended as an operator sign or to '*do something signal*' and not as a symbol of a static relation (Kieran 1981). Interpretation of the trivial equation  $x = 2$  specially points to this direction. For some teachers the statement is only an answer and not an equation, that is, the expression on the left-hand side is a process, which is already performed, whereas the expression on the right-hand side must be a result or an answer. The interpretation of the identity  $\cos^2 \alpha + \sin^2 \alpha = 1$ ; '*It is a result of something*' (Anna) also indicates that the equals sign is used in the meaning 'that is', which is usual in arithmetic. Similarly, one of the teachers compares the statement  $e^{x+y} = 1$  with the trivial equation  $x = 2$  and establishes that it is '*not an equation... it is only an answer*' (Eric).

The statement  $x = 2$  gives for the teachers the feeling that '*x is already solved*' or that it is '*an answer*'. The interpretation of the trivial equation indicates that some teachers may have developed a '*pseudostructural conception*' of equations, that is, they are able to perform

meaningless procedures without having a deeper structural understanding of the concept. (cf. Sfard & Linchevski 1994).

Some teachers do not interpret the statements like (1)  $V = \frac{4\pi r^3}{3}$ , (2)  $\cos^2 \alpha + \sin^2 \alpha = 1$  and (3)  $x^2 - y^2 = (x - y)(x + y)$  as equations. They apprehend them as rules and formulas. One of the reasons may be that these statements according to general conventions, are called with different names like formulae (the volume of sphere) or the identities (Pythagorean identity, conjugate rule) and therefore cannot be understood as equations in the teachers' mind. These different names also reflect different use of 'letters' or variables and they may evoke different feelings for the teachers (cf. Usiskin 1988, 9; Attorps 2005, 2006).

The results of this investigation also indicate that the teachers are unsure of mathematical symbols like  $f(x)$ ,  $\int f(x)dx$ ,  $y''(x)$ ,  $\sqrt{\quad}$  and solving procedures. For some of the teachers the symbol  $\int f(x)dx$  evoked thoughts of a surface or an area under the curve. This conception may depend on the fact that the symbol evokes in the teachers' mind a feeling of a concrete situation related to a definite integral and an area calculation. Structural understanding of mathematical concepts refers to the ability to identify a concept containing many details at 'a glance' (cf. Sfard 1991).

The results in this investigation also indicate that the teachers have difficulties to accept that two quantities (expressions) are equivalent. For instance, one of the teachers wonders, when looking at the statement  $x^2 - y^2 = (x - y)(x + y)$ : *'Is it always an equation, if there is an equals sign?'* (Jenny)

All the findings in this investigation point out the same direction that operational outlook in algebra is fundamental and that the structural approach does not develop immediately (cf. Sfard 1991).

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