UNIVERSITY STUDENTS LINKING LIMITS, DERIVATIVES, INTEGRALS AND CONTINUITY

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Mathematical concepts are mentally represented differently depending on individual, context and existing conceptions of related concepts among other things. The present paper reports on a study of students' representations in analysis with an emphasis on the types of representations and the links they have between their representations. The data collection was designed to evoke different parts of the students' concept images and also to return to the concepts several times over time at every data collection session. The results show that formal and intuitive representations in combination are rare. The number of links between concepts is not in itself a measure of the quality of the concept image, as there is a vast number of erroneous links misleading the students to think they understand the concepts.

INTRODUCTION

Students' experiences of understanding a mathematical concept have a range from being able to explain all aspects of the concept in relation to other concepts (as defined by for example Hiebert and Lefevre, 1986) to just having heard of the concept, depending on how understanding is assessed and personal definitions of understanding. Prior research reveals university students' unjustifiably strong self confidence about their own mathematical abilities of understanding limits of functions (Juter, 2006) despite their inabilities to explain core features of the concept. The capability to solve routine tasks gives a sense of mastery of the concept that is not changed from an episode of failure in a special case, like the interview sessions in the study. A sense of understanding, false or otherwise, prevents further learning in the particular topic area, which in turn may lead to a weak mathematical ground for new learning. This is serious for future mathematics teachers who are going to provide opportune learning situations for their students. If their concept images (Tall & Vinner, 1981) are weak, or in worse cases wrong, there is no room for the flexibility and deep conceptual discussions necessary for appropriate teaching. This paper reports part of a study of pre-service teachers' understanding of limits, derivatives, integrals and continuity and links between the concepts. The study also concerns teacher identity from a social as well as a cognitive perspective (se for example Juter, 2010). The following research questions were raised: How do the students connect limits, derivatives, integrals and continuity to other concepts? How do the students represent (graphically, formally, through examples or other descriptions) the concepts for themselves? How do their representations with connections work as a base for analysing graphs with respect to the four concepts examined?

Data analysis resulted in a classification system (table 1) useful for categorising students' traces of connections between concepts in their concept images in form of mathematics actions such as problem solving, proving and explaining solutions or methods.

UNDERSTANDING AND REPRESENTING CONCEPTS

Skemp (1976) distinguished understanding a concept from its core features, *relational understanding*, which enables implementation of the new concept to existing concept images (which is how Hiebert and Lefevre, 1986, defined conceptual knowledge, p 3), from understanding by just being able to perform a particular operation in what he denoted *instrumental understanding*. Either way to understand a new concept requires mathematical development of existing representations. Tall (2004) introduced a model describing development in three different modes, *the conceptual-embodied world* with an emphasis on exploring activities, *the proceptual-symbolic world* focusing the dual features of concepts as objects and processes expressed in symbols or *procepts* (Gray & Tall, 1994), and *the formal world* where mathematical properties are deduced from the formal language of mathematics in definitions and theorems. Students' concept images develop through the worlds with different emphasis on the three modes allowing them to understand concepts differently. Based on Pinto's and Tall's (2001) definitions of *formal learner* and *natural learner* together with Tall's three worlds and Skemp's definitions of understanding, I created a set of categories to classify students' links between concepts presented in table 1 (see Juter, 2009, 2010, for further details). Examples of classifications of students' links from the present study are provided in the table.

The last four types of links are not desirable for the students, who often are unaware of the quality of the links, particularly if irrelevant or invalid links are mixed with valid ones. Links are formed in different situations, e.g. at lectures, with peers or in solitude. Textbooks, lecturers' selections and general interests of the group of students frame the learning environment and therefore affect the representations students are using. Students' abilities, ambitions and confidence also influence their representations. Representations used when learning a certain topic may become vague if they are not endurable enough, e.g. not sturdily linked to other concepts. If a person learns a new mathematical topic in the embodied world and his/her abilities then develop to symbolic treatment he/she has changed the way of thinking to a proceptual-symbolic mode (Tall, 2008). If the learning phase in the conceptual-embodied world has been too short or otherwise inadequate, parts of the concept image may become disjoint or vague, rendering the person unable to explain core features of the concept. In the present study all students are future mathematics teachers, but they are taught at different universities, by different teachers and under various circumstances which gives them a range of various learning environments to develop mathematical representations from.

Type of link	*Definition*
Valid link, procedural (vp)	True relevant link with focus on calculations or applications, ex: *The derivative of velocity gives acceleration*
Valid link, naturally conceptual (vn)	True relevant link revealing a core feature of the concept, not formal, ex: *Derivative is the slope of the tangent at a point*
Valid link, formally conceptual (vf)	True relevant link formally revealing a core feature of the concept, ex: *If the limit* $\lim_{x \to a} f(x) = f(a)$ *exists in every point then f(x) is continuous*
Irrelevant link, no reason (ir)	No actual motivation for the link is provided
Irrelevant link, no substance (is)	Peripheral true link without substance relevant for the concept, ex: *You can add derivatives*
Invalid link, misconception (im)	Untrue link due to a misconception of the concept, ex: *Continuous means the same change everywhere*
Invalid link, counter perception (ic)	Untrue statement contradicting prior statements ex: *sinx is continuous and continuous means linear*

Table 1. Definitions of links between concepts. Examples in italics.

THE STUDY

Students from four groups, two from each of two different universities in Sweden, were part of the study. All students, a total of 42, were pre-service teachers in mathematics who were studying to teach grades 7 to 9 and upper secondary school. The study started with one group, group 1, two years before the remaining three groups were added to the project as table 2 indicates. All students in the four groups were enrolled in the study.

Group	*Autumn 1*	*Autumn 2*	*Autumn 3*	*Spring 4*	*Autumn 4-5*
1	Questionnaire Tasks	Interview 1	Interview 2	Interview 3 observation	
2-4			Questionnaire Interview 1	Interview 2	Interview 3 observation

Table 2: Data collection times for the groups in years 1 to 5

The first data collection in three of the groups (1, 2 and 4) was at the beginning of the students' analysis course where they filled out a questionnaire aimed at revealing their pre-knowledge of the concepts investigated. The questions were openly formulated for the students to be able to answer without influence from other formulations. Questions about the concepts were "Describe the concept of limit of a function/derivative/integral", "What do you use limits/derivatives/integrals for?" and "What does it mean for a function to be continuous?" There were some questions about the course and a mathematics teacher's main qualities as well. The students' responses to the questionnaires were used to determine which students to interview. The selection was done to reflect the variety of representations, e.g. formal or intuitive, in the descriptions of concepts among the entire group of students. The purpose of the questionnaire in group 3 was the same but it was somewhat differently designed since the students had completed their analysis course at the time of data collection. The questions about what to use the concepts for was replaced by some of the tasks the other groups got after the course. The questions and tasks in the questionnaires and interviews were designed to make the students respond to the four concepts in various contexts and to come back to the same concept repeatedly. This way the responses were confirmed and clarified and different contexts evoked different parts of the students' concept images allowing a more nuanced picture from the data. All interviews were individually conducted and audio recorded. Data was collected from a teacher identity point of view as well, and methodology and results from that part are presented in Juter (2010). Focus in what follows is on the cognitive representations of the students.

The aim with the first interview in groups 1 and 3 was to investigate the students' representations of the concepts as traces of their concept images, and links among the concepts. The questions and tasks were quite open at first to let the students choose their own formulations of the concepts. Then the instruments used were more directed to different aspects of the concepts. One instrument was a table of 29 words and phrases used with the purpose to work as stimuli for the students to recall parts of their concept images. The words were for example tangent, border, sum, slope, rate of change, infinity and interval (see Juter, 2009 for details). The students were to describe the evoked parts or say if there was no recollection linked to a particular concept (results in table 3). This matrix was used in all groups about half year to a whole year after their analysis course to let the students' conceptions stabilize after the course. Four graphs (figure 1) with different characteristics linked to the four concepts studied were also given to the students to determine whether or not the represented functions have limits, are differentiable, integrable and continuous in every point (results in table 4).

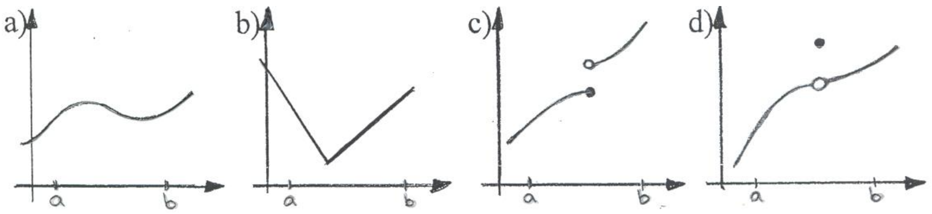

Figure 1. Graphs of functions for the students to analyse

For each of the four concepts, a set of four descriptions was presented to the students on separate cards. The aim was to let the students respond to the different ways of representing the concepts. First they got a formal definition and were asked if they could see which concept it was. Then they got a picture describing the definition which could help them determine which definition it was. After that they got a sentence intuitively explaining the concept and then an example that included calculations related to the concept. The students were asked to choose their preferred representation and also to explain how they would use different representations in their own teaching. They got the concepts in the order limit, derivative, integral and continuity and therefore the last concept could easily be guessed. This was obvious in a few cases when the students were asked to explain the link between the picture and the definition (results in table 5). The first interview in groups 2 and 4 did not include the matrix of recalled links since their analysis courses took place during the same semester. The matrix was presented to those students in the second interview about six months after the analysis courses. The open questions used in the questionnaire were used again in the interviews to reveal how the concepts had developed in the students' concept images. Conceptual issues were addressed again in the second interview in various forms. The first and second interview comprised the same components together for all students, but differently disposed depending on the time of their analysis courses.

RESULTS

The results are divided in three sections according to the data collection with the matrix of words to evoke the students' concept images, the four graphs in figure 1 for the students to analyse, and the cards with various sorts of representations. The analysis was done to reveal the students' perceptions of the concepts, including links between concepts, as traces of their concept images from a range of various mathematical settings. In table 3 the interviewed students' links from the word matrix are classified in the categories defined in table 1. The numbers after the categories determine the number of links of each kind. The students were divided into three groups, I, II and III, according to their conceptions. The three students in the first group, group I, had few valid links of which none were formally conceptual (vf). Several links were irrelevant (ir, is) or invalid (im, ic). The four students in group II had more valid links than the students in group I, but also no formally conceptual ones. The students had few irrelevant or invalid links. Group III comprised three students with many valid links, including formally conceptual links and very few irrelevant or invalid ones. The labels by the students' names in table 3 indicate which group they belong to. The students are listed in order of strength of their concept images with the weakest first.

Student	*Limit*	*Limit*	*Derivative*	*Derivative*	*Integral*	*Integral*	*Continuity*	*Continuity*
Andy (I)	vp 1 vn 2 vf 0	ir 3 is 0 mc 2	vp 5 vn 0 vf 0	ir 3 is 8 mc 1	vp 5 vn 0 vf 0	ir 3 is 8 mc 1	vp 0 vn 0 vf 0	ir 3 is 6 mc 8
Betty (I)	vp 0 vn 1 vf 0	ir 2 is 2 mc 3	vp 1 vn 2 vf 0	ir 1 is 2 mc 1	vp 1 vn 0 vf 0	ir 0 is 0 mc 1	vp 0 vn 1 vf 0	ir 0 is 0 mc 3
Chris	vp 1	ir 5	vp 4	ir 2	vp 5	ir 2	vp 0	ir 1

(I)	vn 1 vf 0	is 1 mc 0	vn 0 vf 0	is 0 mc 3	vn 0 vf 0	is 1 mc 5	vn 3 vf 0	is 0 mc 2
Diana (II)	vp 4 vn 0 vf 0	ir 0 is 1 mc 3	vp 10 vn 0 vf 0	ir 1 is 0 mc 2	vp 5 vn 1 vf 0	ir 0 is 0 mc 2	vp 0 vn 1 vf 0	ir 1 is 1 mc 1
Elise (II)	vp 2 vn 0 vf 0	ir 0 is 0 mc 1	vp 2 vn 5 vf 0	ir 3 is 0 mc 0	vp 4 vn 3 vf 0	ir 1 is 2 mc 0	vp 0 vn 1 vf 0	ir 0 is 2 mc 1
Frank (II)	vp 6 vn 1 vf 0	ir 0 is 2 mc 1	vp 7 vn 2 vf 0	ir 0 is 0 mc 0	vp 3 vn 3 vf 0	ir 1 is 1 mc 0	vp 2 vn 1 vf 0	ir 2 is 0 mc 3
Glenn (II)	vp 7 vn 2 vf 0	ir 2 is 1 mc 5	vp 7 vn 4 vf 0	ir 0 is 3 mc 0	vp 5 vn 0 vf 0	ir 0 is 2 mc 0	vp 4 vn 1 vf 0	ir 0 is 4 mc 2
Harry (III)	vp 7 vn 0 vf 1	ir 0 is 1 mc 2	vp 11 vn 1 vf 0	ir 0 is 2 mc 0	vp 10 vn 0 vf 1	ir 0 is 1 mc 0	vp 3 vn 2 vf 1	ir 0 is 2 mc 1
Ivan (III)	vp 8 vn 1 vf 2	ir 3 is 2 mc 1	vp 6 vn 7 vf 1	ir 5 is 0 mc 0	vp 6 vn 4 vf 1	ir 3 is 3 mc 0	vp 2 vn 4 vf 0	ir 2 is 2 mc 0
John (III)	vp 5 vn 5 vf 3	ir 2 is 1 mc 0	vp 6 vn 1 vf 6	ir 1 is 1 mc 0	vp 7 vn 5 vf 1	ir 0 is 1 mc 0	vp 3 vn 1 vf 0	ir 3 is 1 mc 0

Table 3. Links between concepts categorised according to table 1. The categories Invalid link, misconception (im) and counter perception (ic) are merged in mc

Some students readily talk about their conceptions and views while others are not so forward in an interview situation. It is therefore important to take the total number of links for each student into account when reading table 3. Andy had a large number of irrelevant or invalid links but few valid ones, particularly for continuity where he had no relevant links. He had a lot to say about the concept, for example that a continuous function has to be linear, has no peaks in its graph and changes the same way everywhere, but nothing substantially valid. Andy had a total of 59 links of which 34 were irrelevant and 12 invalid. He showed traces of a stronger concept image for limits than for the other concepts. An example of a valid naturally conceptual link (vn) is his explanation of limits as intervals: "a limit is a form of interval which is shortened to a great extent", and he explained further by drawing a figure of a graph with an interval on the *y*-axis, i.e. the function values, around a point and saying "it [the graph] closes in on the point from both ways and you press together like this [the endpoints of the interval at the *y*-axis]". Other parts of the study confirmed his naturally conceptual understanding of limits, i.e. he was able to discern and explain vital aspects of the concept. Elise had 27 links in all of which 17 were valid, i.e. a considerably larger part of the total number of links than Andy had (13 out of 59 valid). She showed a high level of natural conceptual understanding of the concepts of derivative and integral. John had 53 links. He had 10 formally conceptual links of his total of 43 valid links, leaving 10 irrelevant links. John showed an overall strong and rich concept image of the concepts studied through the data collection. Continuity was the concept with the least amount of links for John, but also for the other students, which may come from the fact that derivatives and integrals are a major part of the required upper secondary school courses and limits are therefore to some extent dealt with, but not necessarily continuity. The students have consequently not had the same time to implement continuity to their concept images as the other concepts.

Graph a	***Yes***	***No***
Limit	ABCEFHIJ	DG
Continuous	ABCDEFGHIJ	
Differentiable	ABCDEFGHIJ	
Integrable	ABCDEFGHIJ	
Graph b	***Yes***	***No***
Limit	ABCEFHIJ	DG
Continuous	BDEFHIJ	ACG
Differentiable	ACDFGHI	BEJ
Integrable	ADEFHIJ	BCG
Graph c	***Yes***	***No***
Limit	CFGH	ABDEIJ
Continuous	C	ABDEFGHIJ
Differentiable	DEFIJ	ABCGH
Integrable	CEFI	ABDGHJ
Graph d	***Yes***	***No***
Limit	ABEGI	CDFHJ
Continuous		ABCDEFGHIJ
Differentiable	DEFI	ABCGHJ
Integrable	CEFI	ABDGHJ

Table 4. Students' answers to whether the four graphs have limits, are differentiable, integrable and continuous in the indicated intervals in the graphs in figure 1

Graph *a* was not a problem to most of the students. On the other hand, it was harder for the students to determine whether Graph *b* is differentiable or not. The peak made them confused and seven students answered incorrectly. Andy, Betty, Elise and Ivan responded correctly to all the limit parts of the tasks and in the last two tasks. Elise, Frank and Ivan were all wrong concerning differentiability. The traces of the concept images in table 3 of these students were very different and in this part of the data collection there is no pattern showing who is high achieving and who is not, or who has a strong concept image or not. Betty, Elise and Ivan had the highest number of correct answers in table 4, but table 3 indicates that they belong to different groups (I, II and III respectively). This type of task is not typically what the students had seen in their courses so it was new to all of them. Chris, Diana and Glenn had the highest error rates. Diana and Glenn were both in group II in table 3 implying a rather strong concept image for the concepts except for limits. Both students had three of the four limit tasks wrong in table 4. Andy and John, in each end of table 3, had the same score in table 4.

Table 5 shows how the students represented the concepts from the cards with four different representations, formal, picture, calculated example and sentence, for each of the four concepts.

Students	***Limit***	***Derivative***	***Integral***	***Continuity***
Andy	*P, DC*	*F, P,* $D_P C$	*P, DC*	*P,* $D_u C$
Betty	*P,* $D_P C$	*S, DI*	*P, E, S, DI*	*P, S, DI*
Chris	*P, S, DI*	*F, P, E, S,* $D_P C$	*P, S, DC*	*E, DI*
Diana	*E, DI*	*F, P,* $D_P C$	*P, S, DI*	*P, DI*
Elise	*E, DI*	*F, P, E, S,* $D_P C$	*P, S, DC*	*S, DI*
Frank	*S, DC*	*E, S,* $D_P C$	*S, DC*	*S,* $D_u C$
Glenn	*S, DI*	*P, E, DC*	*P, DI*	*S,* $D_u C$
Harry	*F, P, DC*	*P, E, DC*	*P, E, DC*	*E, S, DC*
Ivan	*F, P, DC*	*F, P, DC*	*F, P, DC*	*S,* $D_P C$
John	*P, DC*	*F, DC*	*P, DC*	*S, DC*

Table 5. Students' preferred representations of the concepts (*F*: Formal, *P*: Picture, *E*: Example, *S*: Sentence) and their abilities to recognise the formal definitions (*DC*: Definition correct, $D_P C$: Definition + picture correct, $D_u C$: Definition correct but unable to explain, *DI*: Definition incorrect)

Table 5 reveals that continuity was the concept hardest to recognise from the definition, followed by limit. The fact that continuity was the last concept addressed helped the students understand which one it had to be. This became apparent when they were asked to explain the formal representation in relation to the picture (those cases are marked $D_u C$ in the table). All students except Betty recognised the definition of derivative and a majority could identify the definition of integral from the card. Derivative was the concept the students had most kinds of representations for. Chris and Elise both used all four representations intertwined. Students who could recognise the formal definitions (*DC* in table 5) often used pictures to represent the concepts. Pictures in combination with formal representations were also common in the cases where the definition was recognised from the formal representations. Students who were unable to determine which formal definition a card represented, but could determine it from the added picture (marked $D_P C$ in the table) often preferred pictures as a representation for the concept. Betty, Frank and Glenn did not use any formal representations in this context. The students are listed in the same order as in table 3 where the first students have weaker concept images and the last ones have stronger concept images. Andy, who is in the first group, is different from the others with his ability to recognise the concepts from their definitions despite his results reported in table 3. Among the students who could not recognise the definitions (*DI* in table 5) sentences were the preferred alternative, and never formal which could be expected. The sentences were intuitively stated and typically chosen as alternatives to formal representations. There were only two cases of formal and sentence representations together, and in both cases all four representations were used (Chris and Elise).

DISCUSSION

The students' links between topics were very diverse and the data collection reported in table 3 gives a picture of continuity not being implemented in their concept images with very few links of all sorts. The other concepts were more elaborated with more links but the links do not only imply understanding. A large number of links may be misleading the students to believe they understand, like in Andy's case, since they are able to talk about the concept. If counter perceptions, which for example Andy and Chris showed evidence of, can be evoked simultaneously the students are more likely to see their flaws than if their misrepresentations are just invalid without a contradiction.

Andy and Betty represented the concepts mainly in pictures (table 5) and that is reflecting how they read the graphs in table 4 as well. Andy and Betty were among the students with the highest rates of correct answers when analysing the graphs (table 4) and the ones with the weakest concept images in the matrix of words (table 3) which is a surprising result. They were often unable to explain their claims correctly, which is a problem especially for future teachers, but they had a sense of the characteristics of the concepts. Ivan, one of the students with the strongest concept images in table 3, had three errors concerning differentiability in the analysis of the graphs (table 4). Ivan's representations in table 5 are mainly formal and by pictures, moreover table 3 reveals that the derivative is in fact a concept he can handle procedurally, conceptually and formally, yet table 4 shows that he is uncertain of the core features of the concept. This type of overarching non-routine tasks requires the students to know more than just definitions in isolated mathematical contexts. They need to be able to relate the characteristics of the concepts to the graphs with their different properties and to use their representations in their analysis. A strong concept image is not enough for students to discern the necessary features. Comparing with Andy and Betty, the similarities are pictures as preferred representations. Dissimilarities are that Andy and Betty do not use any formal representations in either of the contexts reported in table 3 and 5 except one in Andy's case, whereas Ivan does in both. Ivan has an overall richer representation of all concepts studied and uses and refers to formal formulations, but it appears as if he uses his formal representations without having a clear and flexible understanding of them linked to his other representations of the concepts in situations new to him.

Table 3 and table 5 match each other when it comes to the students' use of formal representations and they indicate that a strong concept image means formal representations as opposed to a weaker concept image. Table 4 reveals a complete lack of trends in relation to tables 3 and 5 and the students' representations reported therein. The traces from their concept images depicted in these three tables imply complex relations between abilities to make assumptions about graphs from the various sorts of representations investigated. Rich concept images, including formal representations, do not imply higher abilities to distinguish the essence of the four concepts from the type of tasks used (figure 1).

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