STUDENTS' CONCEPT DEVELOPMENT OF LIMITS

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Students' learning developments of limits were studied in a calculus course. Their actions, such as problem solving and reasoning, were considered traces of their mental representations of concepts and were used to describe the developments during a semester. Several students went through the course with a vague conception of limits which in some cases was wrong. A higher awareness about their mental representations' reliability is required.

INTRODUCTION

The concept of limits is an important and basic notion among others in a calculus course. Students learning limits of functions perceive and treat limits differently. Embracing limits of functions demands certain abstraction skills from the students. There are several cognitively challenging issues to deal with, such as understanding the quantifiers' roles in the formal definition or linking formally expressed theory to everyday problem solving. Students accept different levels of understanding as they have different priorities and abilities. A study on students' conceptual development of limits of functions was conducted at a Swedish university (Juter, 2006a) with the aim to describe students' developments as they learned limits in a basic calculus course. The results imply differences in high achieving and low achieving students' work with limits, but also a lack of differences at some points as will be discussed further on in this article.

A MODEL OF CONCEPT REPRESENTATIONS

Tall (2004) has introduced three worlds of mathematics to distinguish different modes of mathematical thinking, with the purpose to "gain an overview of the full range of mathematical cognitive development" (Tall, 2004, p. 287). The theory of the three worlds emphasizes the construction of mental representations of concepts and has emerged from several theories on concept development, such as Sfard's (1991) work on encapsulation of processes to objects and Piaget's abstraction theories (Tall, 2004). The three worlds are somewhat hierarchical in the sense that there is a development from just perceiving a concept through actions to formal comprehension of the concept. The first world is called the embodied world and here individuals use their physical perceptions of the real world to perform mental experiments to build mental conceptions of mathematical concepts. The mental experiments can be children's categorisations of real-world objects, such as an odd number of items or, later, students' explorations of intuitive perceptions of limits of functions. The second world is called the proceptual world. Here individuals start with procedural actions on mental conceptions from the first world, as counting, which by using symbols become encapsulated as concepts. The symbols represent both processes and

concepts, for example counting and number or addition and sum. The symbols, together with the processes and the concepts, are called procepts (Gray & Tall, 1994) and are used dually as processes and concepts depending on the context. The third world is called the formal world and here properties are expressed with formal definitions as axioms. There is a change from the second world with connections between objects and processes to the formal world with axiomatic theories comprising formal proofs and deductions. Individuals go between the worlds as their needs and experiences change and mental representations of concepts are formed and altered.

Not all mathematical concepts can be regarded as an object and a process, e.g. a circle or an equivalence class that are both pure objects, though in limits this duality is very obvious. Limits can be handled through an explorative approach with tables of function values and graphs from the beginning and later as symbolically expressed entities. Learning limits of functions demands leaping between operational and static perceptions (Cottrill et al., 1996). There is a challenge in understanding the termination of an infinite procedure as a finite object, such as $\lim\limits_{x\to\infty}\dfrac{\sin\left(\dfrac{1}{x}\right)}{\dfrac{1}{x}}=1$. It is important to reach all significant stages and be able to change between the different stages. Only then can an individual fully understand the concept if understanding of a mathematical concept is defined as Hiebert and Carpenter did (1992), i.e. to be something an individual has achieved when he or she can handle the concept as part of a mental network. The more connections between the mental representations, the better the individual understands the concept (Dreyfus, 1991; Hiebert & Carpenter, 1992).

In an attempt to create a model for concept development, I have used theories about concept images (Tall & Vinner, 1981; Vinner, 1991) as a complement to the theory of the three worlds. A concept image for a concept is an individual's total cognitive representation for that concept. The concept image comprises all representations from experiences linked to the concept, of which there may be several sets of representations constructed in different contexts that possibly merge as the individual becomes more mathematically mature. Multiple representations of the same concept can co-exist if the individual is unaware of the fact that they represent the same concept. Possible inconsistencies may remain unnoticed if the inconsistent parts are not evoked simultaneously. Concept images are created as individuals go through the developments represented by the three worlds. The model in Figure 1 shows how part of a concept image can be structured as I consider it. The three types of symbols used each represent a concept at the stage of one of Tall's three worlds, as described in the figure. The concepts can be, for instance, geometric series, derivatives of polynomials, definitions of derivatives and limits of functions, theorems, proofs, and examples of topics of related concepts. More links and more

representations of concepts exist around the formal world representations of concepts. There are also parts that are not very well connected to other parts. This situation can occur when individuals use rote learning as they try to cope with mathematics. Students who are unable to encapsulate processes as objects or take the step from procepts to a strictly formalistic exposition can use rote learning as a substitute.

Concept images' magnitudes vary according to the topics of the concept image. A concept image of calculus comprises several sub-levels which can be addressed depending on current circumstances. Parts of concept images can, for that reason, be depicted in terms of *topic areas* as a means to talk about different parts of the concept image as shown in Figure 1. The topic areas are themselves concept images used to divide the larger concept image in different areas of mathematics in components of certain topics, e.g. 'functions' or 'limits'. The sizes of topic areas vary according to what context they appear in, for instance large areas such as 'functions', or smaller areas such as 'polynomial functions'. The classification in topic areas means sub-topic areas at several levels. A component in one topic area can in itself be a topic area. Weierstrass's limit definition belongs to the topic area of 'limit', as do 'limits of rational functions' and the symbols used to express limits. The symbols also belong to the topic areas 'derivatives' and 'continuity'. Topic areas overlap this way as illustrated by the simplistic model in Figure 1.

If a concept is represented in more than one topic area in a concept image and the topic areas the representations belong to are disjoined, then inconsistencies may occur in the way aforementioned. Inconsistencies can appear within a topic area as well, but they are easier to detect due to the relatedness of the topic. The development of concept images never ends and the mental representations generate a dynamical system linked together at various levels.

An example of a topic area, marked by a wider contour line in Figure 1 represents the topic area 'limits'. It comprises a marked oval component representing the limit definition, which is also part of the topic areas 'derivatives', TA2, and 'continuity', TA3. The black rectangular component represents the definition of derivatives. The figure only shows some nodes in each topic area to describe the structures of the complicated relations. There are, in most real cases, more nodes linked in more intricate constellations.
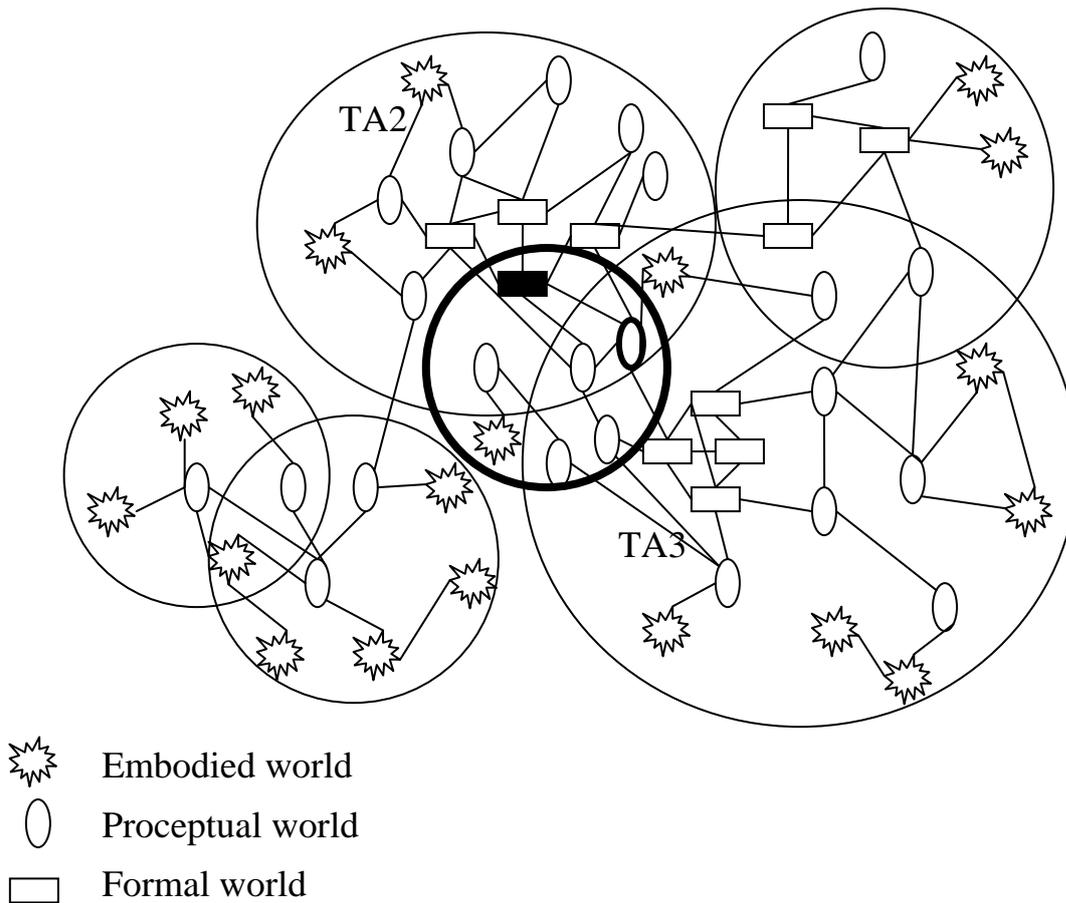
Figure 1. Topic areas (parts of a larger concept image) and components with links in a model of a concept image. The marked part is the topic area 'limits'. TA2 and TA3 represent the topic areas 'derivatives' and 'continuity' respectively.

Concept images change on account of outer and inner stimuli, such as discussions, thoughts and problem solving, and a model such as the one in Figure 1 is hence in constant change. It is nevertheless a tool suitable for describing students' concept developments of limits of functions.

THE EMPIRICAL STUDY

This section describes the sample of students studied and the course they were enrolled in, followed by an outline of methods and instruments used.

The students and the course

There were 112 students participating in the study, of these, 33 were female. The students were aged 19 and up. They were enrolled in a first level university course in mathematics that was divided into two sub-courses. Both of them dealt with calculus and algebra and were given over 20 weeks full time (10 weeks for each course). The students had two lectures (the whole group with one lecturer) and two sessions for task solving (in sub-groups of 30 students with a teacher in each sub-group) three days per week. Each lecture and session lasted 45 minutes. Thus the total teaching time for each course was 90 hours.

The first course had a written exam and the second had a written exam followed by an oral one. The marks awarded were *IG* for not passing, *G* for passing and *VG* for passing with a good margin.

Methods and instruments

Different methods were used to collect different types of data, such as students' solutions to limit tasks and responses to attitudinal queries. The sets of data were collected at different stages in the students' developments to describe traces of changes in their concept images during the semester. The instruments used were designed to take those differences into account. The limit tasks were of increasing difficulty and the attitudinal part was mainly in the beginning of the semester. The students were confronted with tasks at five times during the semester, called *stage A* to *stage E*.

The students got a questionnaire at stage A in the beginning of the semester. It contained easy tasks about limits, such as:

Example 1: $f(x) = \dfrac{x^2}{x^2 + 1}$. What happens with f(x) if *x* tends to infinity?

The tasks did not mention limits per se, but were designed as a means to explore if the students could investigate functions with respect to limits. The students were also asked about the situations in which they had met the concept before they started their university studies to reveal if they had related topic areas linked to the topic area 'limits of functions'. Some attitudinal data was also collected.

After limits had been taught in the first course the students received a second questionnaire at stage B, with more limit tasks at different levels of difficulty, for example:

Example 2: a) Decide the limit: $\lim\limits_{x \to \infty} \dfrac{x^3 - 2}{x^3 + 1}$. b) Explanation.

c) Can the function $f(x) = \dfrac{x^3 - 2}{x^3 + 1}$ attain the limit value in 2a? d) Why?

The aim was for the students to reveal their habits of calculating, their abilities to explain what they did, and their attitudes in some areas. The students were asked if they were willing to participate in two individual interviews later that semester. Thirty-eight students agreed to do so; of these, 18 students were selected for two individual interviews each. The selection was done with respect to the students' responses to the questionnaires so that the sample would as much as possible resemble the whole group. The gender composition of the whole group was also considered in the choices.

The first session of interviews was held at stage C in the beginning of the second course. Each interview was about 45 minutes long. The students were asked about definitions of limits, both the formal one from their textbook and their individual

ways to define a limit of a function. The students' topic areas 'limits in theory' and 'limits in problem solving' were investigated with the purpose to discover possible links or inconsistencies. Example 3 was used for that reason:

Example 3: Is it the same thing to say "For every $\delta > 0$ there exists an $\varepsilon > 0$ such that $|f(x) - A| < \varepsilon$ for every x in the domain with $0 < |x - a| < \delta$" as "For every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - A| < \varepsilon$ for every x in the domain with $0 < |x - a| < \delta$"? What is the difference if any?

They were also asked to comment on statements very similar to those used by Williams (1991) in a study about students' models of limits. The statements the students commented on are the following (translation from Swedish):

1. A limit value describes how a function moves as x tends to a certain point.

2. A limit value is a number or a point beyond which a function can not attain values.

3. A limit value is a number which y-values of a function can get arbitrarily close to through restrictions on the x-values.

4. A limit value is a number or a point which the function approaches but never reaches.

5. A limit value is an approximation, which can be as accurate as desired.

6. A limit value is decided by inserting numbers closer and closer to a given number until the limit value is reached.

The reason for having these statements was to get to know the students' perceptions about the ability of functions to attain limit values and other characteristics of limits. Connections to language and intuitive representations were discussed to some extent to further describe their concept images. The students also solved limit tasks of various types with the purpose to reveal their perceptions of limits and commented on their own solutions from the questionnaires to clarify their written responses where it was needed.

The students received a third questionnaire at the end of the semester, at stage D. It contained just one task. Two fictional students' discussion about a problem was described. One reasoned incorrectly and the other one objected and proposed an argument to the objection. The students in the study were asked to decide who was correct and why.

A second interview was carried through at stage E after the exams. Each interview lasted for about 20 minutes. Of the 18 students, 15 were interviewed at this point. The remaining three students were unable to participate for various reasons. The students commented on the last questionnaire and, linked to that, the definition was scrutinized again. The task at stage D and Example 3 was brought up again to reveal any changes in perceptions. The quantifiers *for every* and *there exists* in the ε - δ definition were discussed thoroughly.

Field notes were taken during the students' task solving sessions and at the lectures when limits were treated to give a sense of how the concept was presented to the students and how the students responded to it. Tasks and results from other parts of the study are described in more detail in other articles (Juter, 2005a-2006c).

THE STUDENTS' CONCEPTUAL DEVELOPMENT

The students' responses to tasks and questions in the questionnaires and interviews have been analysed and categorised. Table 1 shows developments of four of the students with conceptual developments typical for a group of students. The numbers in brackets after each name in the table indicates the number of students in each group. Pseudonym names have been used to retain student anonymity. The digits at stage C indicate the students' preferred choices from the six statements. The bold and larger digits are the students' choices of statements most similar to their own perceptions of limits. The first points at stage E are the students' explanations to what a limit is and the last points are about Example 3 where the students explain the difference between the correct and incorrect definition connected to the task from stage D.

Table 1: Student developments at the five stages A-E

TIME	**Tommy** (3)	**Leo** (3)	**Mikael** (4)	**Julia** (2)
A	-Links to derivatives -Solves easy tasks well	-Links to nutrition, physics and biology -Solves easy tasks well	-Links to prior studies, problem solving, physics -Unable to solve easy tasks	-Links to prior studies -Solves easy tasks well
B	-Limits are attainable in problem solving -Can not state a definition -Solves routine tasks and explains	-Limits not attainable in problem solving -Can not state a definition -Solves routine tasks and explains	-Limits are not attainable in problem solving -Can not state a definition -Solves tasks and explains well	-Limits are attainable in problem solving -Can state a definition -Solves tasks and explains fairly well
C	-Limits are attainable in problem solving -Limits are not attainable in theory -A limit of a function is how the limit stands with respect to another function, no motion -3, **4**, 5, 6 -Can not state or identify the definition	-Limits are not attainable in problem solving (hesitates) -Limits are not attainable in theory -Thinks of limits in pictures -3, **4** -Can not state or identify the definition -Solves tasks well -Links to derivatives and number	-Limits are attainable in problem solving -Limits are not attainable in theory -Thinks logically rather than explicitly define -2, 3, **4**, 5 -Can not state but can identify the definition after investigation -Problems to solve tasks -Links to prior studies	-Limits are attainable in problem solving -Limits are attainable in theory -It comes closer and closer to A as x comes closer and closer to a -1, **3**, 5 -Can not state but can identify the definition (claims that both def in ex 3 state the same, ε and δ come

	-Not a real limit if attainable by function -Problems to solve tasks -Links to derivatives and curves -Claims to master the notion of limits	sequences -Claims to master the notion of limits fairly well	and graphs -Claims to master the notion of limits in problem solving but not the definition	in pairs) -Solves tasks well -Links to derivatives, graphs and number sequences -Claims to master the notion of limits
D	-Identifies the error	-Identifies the error, makes other error	-Identifies the error, makes other error	-Identifies the error
E	-A limit is a point to tend to but never to reach, motion -Links to integrals -Can not identify the definition -Does not understand the quantifiers' roles	-A function tends to a value as the variable controlling the function changes, it never reaches the limit -No links -Can not identify the definition -Does not understand the quantifiers' roles	-If the x-value tends to a value then the function tends to a value connected to the x-value -Links to Taylor expansions, series and number sequences -Can identify the definition -Can explain the difference between the definitions in ex 3	-If the function goes closer and closer a number as x goes closer and closer to a value -Links to continuity -Can identify the definition after investigation -Can explain the difference between the definitions in ex 3 fairly well

The conceptual development as traces of concept images in Tommy's case shows that when he came to the university, he had a concept image of limits of functions linked to derivatives. Limits were treated with a strong focus on problem solving developing the topic area 'limits in problem solving' but not 'limits in theory', and Tommy thought that functions can attain limit values when he dealt with problem solving but not in theoretical discussions for the duration of the course. Tommy did not grasp the notion of limits, but he was able to solve routine tasks. He was confident about his own abilities to master the notion though he did not reach the third of Tall's worlds (Tall, 2004).

The development in Leo's case differed at some points from Tommy's. Leo thought that limits were never attainable, neither in problem solving nor in theoretical situations. Leo was a better problem solver than Tommy was but he too had theoretical problems.

Mikael had trouble to solve tasks at first, but he showed trace of a concept image of limits in connection to derivatives. He thought that limits are unattainable in problem solving at stage B but changed his mind at stage C. He thought that limits are unattainable in theoretical discussions. He had a reasoning approach to mathematics and managed to investigate mathematical situations to find answers. At the end of the course, he was able to explain the theoretical issues correctly. Mikael showed traces of progression through Tall's three worlds from an exploring attitude to a formal understanding.

Julia understood the notion of limits from the start. She had a concept image from upper secondary school when she came to the university. There were some theoretical difficulties during the semester, but they were overcome at the end.

The quantifiers in the definition caused confusion for all students. There was an opinion among some students that ε and δ in Example 3 at stage C come in pairs and can therefore be placed either way in the example. Mostly high achieving students shoved traces of this conception, which can be explained by the fact that low achieving students had not integrated the theory well enough in their concept images to even identify the definition next to a wrong one. The high achieving students did not have this misunderstanding at stage E as they were able to explain the meaning of the quantifiers. The low achieving students did not understand the quantifiers meaning in the definition throughout the course.

The students' problems to connect theory to problem solving became particularly apparent from their difficulties to determine whether limits are attainable for functions or not. Many students interpreted the strict inequalities in the formal definition to say that limits are not attainable. Examples where limits were attainable did not change the low achieving students' beliefs about the definitions' meaning. Some students became frustrated when they saw examples of attainable limits and were asked questions about the definition because they were unable to create a coherent picture of the situation. The students' concept images were divided in disjoint topic areas; 'limits in theory' and 'limits in problem solving'.

Learning limits requires skills from many mathematical areas. Students need to be able to understand formal expositions, perform algebraic manipulations, understand the meanings of quantifiers and absolute values, which students found problematic, and link theory to their every day problem solving. This means that their concept images need to be well developed both in depth and width, i.e. they need abstraction skills and strong links between numerous topic areas. They also need to find inspiration and reasons to go through the hard work to make the knowledge meaningful in their concept images.

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